MEMOIRS
OF
THE INDIAN INSTITUTE OF SCIENCE

Vol. IV, No. 18

RING-SINGULARITY IN BORN'S UNITARY THEORY—I.

BY
B. S. MADHAVA RAO.

FROM THE "PROCEEDINGS OF THE INDIAN ACADEMY OF SCIENCES"
VOL. IV

Reprinted from "The Proceedings of the Indian Academy of Sciences", Vol. IV, No. 3, Sec. A, 1936.

RING-SINGULARITY IN BORN'S UNITARY THEORY—I.

By B. S. Madhava Rao.
(University of Mysore.)
(From the Department of Physics, Indian Institute of Science, Bangalore.)

Received August 17, 1936.

1. Introduction.

In two papers on the unitary theory of field and matter Born[1] has given a classical treatment of the derivation of the equations of motion of a point-singularity representing a particle. The chief assumption made is that the external field must be constant over the "diameter" of the particle. The method is to start from a variation principle representing both the motion of the field and the singularity, the latter part also being written as a space-time integral involving Dirac's δ-function. In addition Born has also introduced a spin for the particle and given a derivation of Kramers'[2] formulæ on the basis of the unitary theory. A treatment on similar lines without however introducing the spin has also been given by Pryce[3] who starts with the Hamiltonian instead of the Lagrangian.

In the second of the papers referred to above, Born has shown that the results obtained by Kramers are difficult to interpret on the basis of the unitary theory and shown that the difficulty disappears if the particle be assumed to have, besides the magnetic moment, a charge or an electric moment or both. This enables the derivation of a set of equations of motion which are a generalisation of Kramers' equations, self-consistent without imposing a restriction on the Lagrangian and satisfactory from the purely formal standpoint. Since, however, the elementary particles occurring in nature have no electric rest-moment the conclusion was drawn that, on the basis of the unitary theory, point-singularities were not the correct representation of the particles. It was finally suggested that there might be other possibilities.

I have considered in this paper an elementary particle as a ring-singularity. The principal results obtained can be summarised as under :—

(*a*) Finite expressions are obtained for the total energy and angular momentum.

---

1 M. Born, *Proc. Ind. Acad. Sci.*, A, 1936, 3, 8; *ibid.*, 1936, A, 3, 85.

2 H. A. Kramers, *Physica*, 1934, 1, 825; Zeeman, *Verhandelingen*, 1935, p. 403.

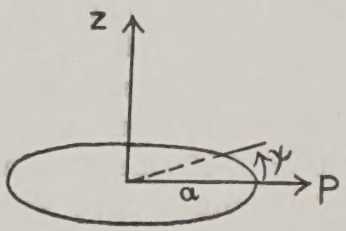
3 M. H. L. Pryce, *Proc. Roy. Soc.*, A, 1936, **155**, 597.

(*b*) The ratio of the magnetic moment to angular momentum is obtained as $e/2\mu$ and not e/μ (μ = rest-energy) showing that we do not obtain an explanation of the spin.

(*c*) Taking the ring-singularity as representing a proton we can derive an estimate of the ring-radius so as to explain the high mass of the proton.

2. *Field Equations.*

Let the axis of symmetry be taken as the z-axis and the radius of the ring a. We introduce cylindrical co-ordinates ρ, ψ, z and assume axial symmetry. The linear density is given by $\eta = e/2\pi a$ and $\vec{v}$ denotes the velocity vector. We introduce the Dirac δ-function such that $\delta = \infty$ on the ring and



$$\int \delta d\rho dz = 1.$$

Proceeding as in the papers of Born quoted above[4] without using spin considerations we obtain the field equations for the stationary case in the form.

$$(2,\ 1) \qquad \begin{cases} \text{rot } \vec{H} = \eta \vec{v} \delta\,; & \text{div } \vec{D} = \eta\delta\,; \\ \text{div } \vec{B} = 0\,; & \text{rot } \vec{E} = 0. \end{cases}$$

We shall now obtain the equations of transformation for vector-components in the case of axial symmetry and also derive expressions for the operators *div*, and *rot*.[5] From the equations of transformation,

$$\left.\begin{aligned} x &= \rho\cos\psi \\ y &= \rho\sin\psi \\ z &= z \end{aligned}\right\} \tan\psi = \frac{y}{x},\ \frac{\partial\psi}{\partial x} = -y/\rho^2 \text{ and } \frac{\partial\psi}{\partial y} = \frac{x}{\rho^2}$$

one has

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{x}{\rho}\frac{\partial}{\partial\rho} - \frac{y}{\rho^2}\frac{\partial}{\partial\psi} \\ \frac{\partial}{\partial y} &= \frac{y}{\rho}\frac{\partial}{\partial\rho} + \frac{x}{\rho^2}\frac{\partial}{\partial\psi} \end{aligned}\right\} \text{ and } \frac{\partial}{\partial\rho}\frac{x}{\rho} = 0,\ \frac{\partial}{\partial\rho}\frac{y}{\rho} = 0$$

and therefore for any scalar ϕ of cylindrical symmetry $\left(\frac{\partial\phi}{\partial\psi} = 0\right)$

$$(2,\ 2) \qquad \frac{\partial\phi}{\partial x} = \frac{x}{\rho}\frac{\partial\phi}{\partial\rho};\ \frac{\partial\phi}{\partial y} = \frac{y}{\rho}\frac{\partial\phi}{\partial\rho}$$

---

[4] See Born, I, p. 10, equations (1, 9).

[5] I have to thank Dr. Beth, one of Prof. Born's pupils, for having communicated to me the correct proofs of the derivation of these expressions and of (2, 6).

and for any vector $\vec{A}$,

$$(2,3)\qquad \left\{\begin{array}{l|l} A_x = \dfrac{x}{\rho}A_\rho - \dfrac{y}{\rho}A_\psi & A_\rho = \dfrac{x}{\rho}A_x + \dfrac{y}{\rho}A_y \\ A_y = \dfrac{y}{\rho}A_\rho + \dfrac{x}{\rho}A_\psi & A_\psi = -\dfrac{y}{\rho}A_x + \dfrac{x}{\rho}A_y \end{array}\right.$$

Accordingly

$$\left.\begin{array}{l} \text{grad}_\rho\ \phi = \dfrac{\partial\phi}{\partial\rho} \\ \text{grad}_\psi\ \phi = 0 \\ \text{grad}_z\ \phi = \dfrac{\partial\phi}{\partial z} \end{array}\right\}$$

so that if a vector be a gradient it has no ψ-component.

To calculate div $\vec{A}$ for any arbitrary vector $\vec{A}$, we divide the vector field into two fields: $\vec{A} = \vec{B} + \vec{C}$, such that $B_\psi \equiv 0$, $C_\rho \equiv 0$, $C_z \equiv 0$, *i.e.*, $|C| = C = C_\psi$. We can prove that div $\vec{C} = 0$, for

$$\text{div}\ \vec{C} = \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \frac{\partial C_z}{\partial z} = \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y}$$

$$\frac{\partial C_x}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{y}{\rho}C\right] = -\frac{x}{\rho}\frac{\partial}{\partial\rho}\left(\frac{y}{\rho}C\right) + \frac{y}{\rho^2}\frac{\partial}{\partial\psi}\left(\frac{y}{\rho}C\right)$$

$$= -\frac{xy}{\rho^2}\frac{\partial C}{\partial\rho} + \frac{y}{\rho^2}C\frac{\partial}{\partial\psi}\frac{y}{\rho}$$

$$= -\frac{xy}{\rho^2} + \frac{xy}{\rho^3}C.$$

$$\frac{\partial C_y}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x}{\rho}C\right] = \frac{y}{\rho}\frac{\partial}{\partial\rho}\left(\frac{x}{\rho}C\right) + \frac{x}{\rho^2}\frac{\partial}{\partial\psi}\left(\frac{x}{\rho}C\right)$$

$$= \frac{xy}{\rho^2}\frac{\partial C}{\partial\rho} - \frac{xy}{\rho^3}C$$

$\therefore\ \dfrac{\partial C_x}{\partial x} + \dfrac{\partial C_y}{\partial y} = 0$, *i.e.*, div $\vec{C} = 0$ and div $\vec{A}$ = div $\vec{B}$;

but, $\text{div}\ \vec{B} = \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} + \dfrac{\partial B_z}{\partial z}$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\rho}B_\rho\right) + \frac{\partial}{\partial y}\left(\frac{y}{\rho}B_\rho\right) + \frac{\partial B_z}{\partial z}, \text{ using (2, 3) since } B_\psi = 0$$

$$= \frac{B_\rho}{\rho} + x\frac{\partial}{\partial x}\left(\frac{B_\rho}{\rho}\right) + \frac{B_\rho}{\rho} + y\frac{\partial}{\partial y}\left(\frac{B_\rho}{\rho}\right) + \frac{\partial B_z}{\partial z}$$

$$= \frac{2B_\rho}{\rho} + x\frac{\partial}{\partial\rho}\left(\frac{B_\rho}{\rho}\right)\frac{\partial\rho}{\partial x} + y\frac{\partial}{\partial\rho}\left(\frac{B_\rho}{\rho}\right)\frac{\partial\rho}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$= \frac{2B_\rho}{\rho} + \rho \frac{\partial}{\partial \rho}\left(\frac{B_\rho}{\rho}\right) + \frac{\partial B_z}{\partial z} \left[\text{since from } \rho^2 = x^2 + y^2 \right.$$

$$\left. x \frac{\partial \rho}{\partial x} + y \frac{\partial \rho}{\partial y} = \rho \right.$$

$$= \frac{B_\rho}{\rho} + \frac{\partial B_\rho}{\partial \rho} + \frac{\partial B_z}{\partial z}$$

Hence, for any arbitrary vector $\vec{A}$

$$(2, 4) \qquad \text{div } \vec{A} = \frac{1}{\rho}\left\{\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{\partial(\rho A_z)}{\partial z}\right\}.$$

We shall now calculate rot $\vec{A}$ for an arbitrary vector $\vec{A}$ which is however axially-symmetric, *i.e.*, for which

$$\frac{\partial A_\rho}{\partial \psi} = \frac{\partial A_\psi}{\partial_\psi} = \frac{\partial A_z}{\partial_\psi} = 0.$$

$$\text{rot}_x \vec{A} = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{y}{\rho}\frac{\partial A_z}{\partial \rho} - \frac{y}{\rho}\frac{\partial A_\rho}{\partial z} - \frac{x}{\rho}\frac{\partial A_\psi}{\partial z}$$

$$\text{rot}_y \vec{A} = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \frac{x}{\rho}\frac{\partial A_\rho}{\partial z} - \frac{y}{\rho}\frac{\partial A_\psi}{\partial z} - \frac{x}{\rho}\frac{\partial A_z}{\partial \rho}$$

$$\text{rot}_z \vec{A} = \frac{x}{\rho}\frac{\partial}{\partial \rho}\left\{\frac{y}{\rho} A_\rho + \frac{x}{\rho} A_\psi\right\} - \frac{y}{\rho^2}\frac{\partial}{\partial_\psi}\left\{\frac{y}{\rho} A_\rho + \frac{x}{\rho} A_\psi\right\}$$

$$- \frac{y}{\rho}\frac{\partial}{\partial \rho}\left\{\frac{x}{\rho} A_\rho - \frac{y}{\rho} A_\psi\right\} - \frac{x}{\rho^2}\frac{\partial}{\partial_\psi}\left\{\frac{x}{\rho} A_\rho - \frac{y}{\rho} A_\psi\right\}$$

$$= \frac{\partial A_\psi}{\partial \rho} + \frac{1}{\rho} A_\psi.$$

Hence,

$$(2, 5) \qquad \begin{cases} \text{rot}_\rho \vec{A} = -\dfrac{\partial A_\psi}{\partial z} \\ \text{rot}_\psi \vec{A} = \dfrac{\partial A_\rho}{\partial z} - \dfrac{\partial A_z}{\partial \rho} \\ \text{rot}_z \vec{A} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho A_\psi) \end{cases}$$

We shall now prove that for the field vectors $\vec{E}$, $\vec{H}$, $\vec{B}$, $\vec{D}$ satisfying the differential equations (2, 1), we have the relations

$$(2, 6) \qquad E_\psi = 0, \quad H_\psi = 0, \quad B_\psi = 0, \quad D_\psi = 0.$$

As regards $\vec{E}$, (2, 1) shows that it is irrotational and has no singularities. Hence it is the gradient of a non-singular scalar and from the theorem proved on p. 357, it follows that $E_\psi = 0$.

Next considering $\vec{H}$, we observe that it has a singularity and cannot be expressed as the gradient of an analytic scalar function. We write

$\vec{H} = \vec{H}_{cl} + \vec{H}^1$, the first term being the classical part

on account of rot $\vec{H} = 0$ and rot $\vec{H}_{cl} = 0$, we get rot $\vec{H}^1 = 0$. Also $\vec{H}_{cl}$ has the same singularities as $\vec{H}$ and accordingly $\vec{H}^1$ has no singularities. This fact along with rot $\vec{H}^1 = 0$ enables us to deduce as above that $\vec{H}^1$ has no ψ-component. But $\vec{H}_{cl}$ is well-known and has no ψ-component. Accordingly $H_\psi = 0$.

The proof for $\vec{B}$ is obtained by considering

$$\vec{H} = \frac{\partial L}{\partial \vec{B}} \text{ where } L = L(F, G) \text{ is the Lagrangian}$$

$$i.e., \quad \vec{H} = L_F \frac{\partial F}{\partial \vec{B}} + L_G \frac{\partial G}{\partial \vec{B}} = L_F \vec{B} + L_G \vec{E}$$

Neither $\vec{H}$ nor $\vec{E}$ has a ψ-component. Hence at all places where $L_F \neq 0$, $\vec{B}$ also has no ψ-component.

A similar proof for $\vec{D}$ is obtained from

$$\vec{D} = -\frac{\partial L}{\partial \vec{E}} = -L_F \frac{\partial F}{\partial \vec{E}} - L_G \frac{\partial G}{\partial \vec{E}} = L_F \vec{E} - L_G \vec{B}$$

$\vec{E}$ has no ψ-component and $\vec{B}$ none when $L_F \neq 0$. Hence $\vec{D}$ has no ψ-component at all places where $L_F \neq 0$ or $L_G = 0$.

Using (2, 4), (2, 5) and (2, 6) we can now write the field equation (2, 1) in the form

$$(2, 7) \qquad \begin{cases} \dfrac{\partial H_\rho}{\partial z} - \dfrac{\partial H_z}{\partial \rho} = \eta v \delta\,; \quad \dfrac{1}{\rho}\left\{\dfrac{\partial(\rho D_\rho)}{\partial \rho} + \dfrac{\partial(\rho D_z)}{\partial z}\right\} = \eta\delta \\ \dfrac{\partial(\rho B_\rho)}{\partial \rho} + \dfrac{\partial(\rho B_z)}{\partial z} = 0, \quad \dfrac{\partial E_z}{\partial \rho} - \dfrac{\partial E_\rho}{\partial z} = 0 \end{cases}$$

since, further, in the stationary case $v_\rho = v_z = 0$, $v_\psi = v$. The Lagrangian is given by

$$(2, 8) \quad L = b^2 \left\{ \sqrt{1 + \frac{1}{b^2}(B_\rho^2 + B_z^2 - E_\rho^2 - E_z^2) - \frac{1}{b^4}(B_\rho E_\rho + B_z E_z)^2} - 1 \right\}$$

3. Boundary conditions at $\rho = a$.

Integrating the field equations over the element of volume

$$\int\int \frac{1}{\rho}\left\{\frac{\partial(\rho D_\rho)}{\partial \rho} + \frac{\partial(\rho D_z)}{\partial z}\right\} 2\pi\rho d\rho dz = \int\int \eta\delta \cdot 2\pi\rho d\rho dz = 2\pi a\eta = e$$

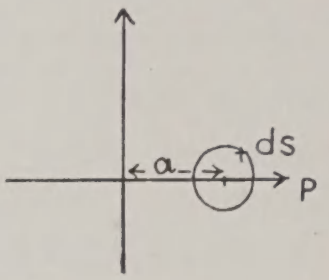
and the left-hand side

$$= 2\pi \int\int \left\{ \frac{\partial(\rho D_\rho)}{\partial \rho} + \frac{\partial(\rho D_z)}{\partial z} \right\} d\rho dz$$

$$= 2\pi \int_0 \rho \{D_\rho \cos(\rho, \nu) + D_z \cos(z, \nu)\} \, ds = 2\pi a \int_0 D_\nu ds$$

(3, 1)

$$\left\{ \begin{array}{l} i.e., \int_0 D_\nu ds = \frac{e}{2\pi a} = \eta \\ \text{Similarly,} \int_0 B_\nu ds = 0 \end{array} \right\}$$



Similarly

$$\int\int \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} \right) 2\pi\rho d\rho dz = \int\int \eta v \delta \cdot 2\pi\rho d\rho dz = 2\pi a v \eta = ev$$

and the left-hand side

$$= 2\pi a \int_0 \{H_\rho \cos(z, \nu) - H_z \cos(\rho, \nu)\} \, ds$$

$$= -2\pi a \int_0 H_s \, ds$$

(3, 2)

$$\left\{ \begin{array}{l} i.e., \int_0 H_s ds = -\eta v \\ \text{Similarly,} \int_0 E_s ds = 0 \end{array} \right\}$$

4. Solution in the case of zero-approximation.

Take $\rho = a$.

In this case the equations

(4, 1) $$\frac{\partial B_\rho}{\partial \rho} + \frac{\partial B_z}{\partial z} = 0 \, ; \; \frac{\partial E_z}{\partial \rho} - \frac{\partial E_\rho}{\partial z} = 0$$

are satisfied by taking $B_\rho = \lambda E_z$ and $B_z = -\lambda E_\rho$, so that

$$B_\rho E_\rho + B_z E_z = 0$$

and the Lagrangian

$$L = b^2 \left\{ \sqrt{1 - \frac{1}{b^2}(1-\lambda^2)(E_\rho^2 + E_z^2)} - 1 \right\}$$

$$= b^2 \left\{ \sqrt{1 + \frac{1}{b^2}\left(1 - \frac{1}{\lambda^2}\right)(B_\rho^2 + B_z^2)} - 1 \right\}$$

or, without making these substitutions

$$L = b^2 \left\{ \sqrt{1 + \frac{1}{b^2}(B_\rho^2 + B_z^2 - E_\rho^2 - E_z^2)} - 1 \right\}$$

From this last form of L,

$$H_\rho = \frac{\partial L}{\partial B_\rho} = \frac{B_\rho}{\sqrt{1 + \frac{1}{b^2}(B_\rho^2 + B_z^2 - E_\rho^2 - E_z^2)}} = \frac{\lambda E_z}{\sqrt{\quad,,}}$$

$$= -\lambda \frac{\partial L}{\partial E_z} = \lambda D_z$$

Similarly $H_z = -\lambda D_\rho$.

These relations could also have been deduced from the relations

$$\frac{\partial H_z}{\partial \rho} - \frac{\partial H_\rho}{\partial z} = 0\,; \quad \frac{\partial D_\rho}{\partial \rho} + \frac{\partial D_z}{\partial z} = 0$$

and we would have obtained $H_\rho = \mu D_z$ and $H_z = -\mu D_\rho$; but, the above method however shows that $\mu = \lambda$.

We can determine λ from the boundary conditions

$$-\eta v = \int_0 H_s ds = \int_0 \{H_z \cos(\rho, \nu) - H_\rho \cos(z, \nu)\}\, ds$$

$$= -\lambda \int_0 \{D_\rho \cos(\rho, \nu) + D_z \cos(z, \nu)\}\, ds$$

$$= -\lambda \int D_\nu ds = -\lambda\eta$$

(4, 2) $\quad \therefore \quad \lambda = v.$ Let $1 - \lambda^2 = 1 - v^2 = a^2$.

Rewriting the equations in the case of the zero-approximation

$$\frac{\partial D_\rho}{\partial \rho} + \frac{\partial D_z}{\partial z} = 0, \quad \frac{\partial E_z}{\partial \rho} - \frac{\partial E_\rho}{\partial z} = 0$$

with the boundary conditions

$$\int_0 D_\nu\, ds = \eta\,; \quad \int_0 E_s\, ds = 0$$

we find that a solution is given by

$$(4, 3) \qquad \left\{ \begin{array}{l} D_\rho = \dfrac{\eta}{2\pi r} \cdot \dfrac{\rho - a}{r} \\ \\ D_z = \dfrac{\eta}{2\pi r} \cdot \dfrac{z}{r} \end{array} \right\}$$

where $r^2 = (\rho - a)^2 + z^2$, *i.e.*, r is the distance from a point on the ring. We can easily express $\vec{E}$ in terms of $\vec{D}$ by considering the Hamiltonian H.

$$H = b^2 \left\{ \sqrt{1 + \frac{1}{b^2}(\vec{D}^2 - \vec{H}^2) - \frac{1}{b^4}(\vec{D}\,\vec{H})^2} - 1 \right\}$$

$$= b^2 \left\{ \sqrt{1 + \frac{1}{b^2}(\vec{D}^2 - \vec{H}^2)} - 1 \right\} \text{ since } (\vec{D}\,\vec{H}) = 0 \text{ in this case,}$$

as easily follows from the relation $H_\rho = \lambda D_z$ and $H_z = -\lambda D_\rho$.

$$\therefore E_\rho = \frac{\partial H}{\partial D_\rho} = \frac{D_\rho}{\sqrt{1 + \frac{1}{b^2}(\vec{D}^2 - \vec{H}^2)}} = \frac{D_\rho}{\sqrt{1 + \frac{a^2}{b^2}(D_\rho^2 + D_z^2)}}$$

and $$E_z = \frac{\partial H}{\partial D_z} = \frac{D_z}{\sqrt{1 + \frac{1}{b^2}(\vec{D}^2 - \vec{H}^2)}} = \frac{D_z}{\sqrt{1 + \frac{a^2}{b^2}(D_\rho^2 + D_z^2)}}$$

Putting in the values of D_ρ and D_z from (1)

$$(4, 4) \qquad \left\{ \begin{aligned} E_\rho &= \frac{\eta}{2\pi r} \cdot \frac{\rho - a}{\sqrt{r^2 + r_1^2}} \\ E_z &= \frac{\eta}{2\pi r} \cdot \frac{z}{\sqrt{r^2 + r_1^2}} \end{aligned} \right\}$$

$$(4, 5) \qquad \text{where } r_1 = \frac{a\eta}{2\pi b}.$$

It is easy to verify that the E-equation, *viz.*, $\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = 0$ is satisfied by the values (2). This needs only showing that

$$(\rho - a)\frac{\partial}{\partial z}\left\{\frac{1}{r\sqrt{r^2 + r_1^2}}\right\} = z\frac{\partial}{\partial \rho}\left\{\frac{1}{r\sqrt{r^2 + r_1^2}}\right\}$$

$$i.e., \quad (\rho - a)\frac{\partial r}{\partial z} = z\frac{\partial r}{\partial \rho}$$

and this is true since $\frac{\partial r}{\partial z} = \frac{z}{r}$ and $\frac{\partial r}{\partial \rho} = \frac{\rho - a}{r}$.

From (2) $E = \frac{\eta}{2\pi} \cdot \frac{1}{\sqrt{r^2 + r_1^2}}$ and is finite for $r = 0$, having the value

$$(4, 6) \qquad \frac{\eta}{2\pi r_1} = \frac{b}{a} = b'.$$

We can now complete the solutions in the case of zero-approximation

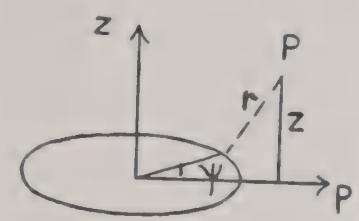
$$(4, 7) \qquad \left\{ \begin{aligned} H_\rho &= \frac{\eta v}{2\pi r} \cdot \frac{z}{r} \\ H_z &= -\frac{\eta v}{2\pi r} \cdot \frac{\rho - a}{r} \end{aligned} \right\}$$

and

$$(4, 8) \qquad \left\{ \begin{aligned} B_\rho &= \frac{\eta v}{2\pi r} \cdot \frac{Z}{\sqrt{r^2 + r_1^2}} \\ B_z &= -\frac{\eta v}{2\pi r} \cdot \frac{\rho - a}{\sqrt{r^2 + r_1^2}} \end{aligned} \right\}$$

5. Solutions in the classical case.

Treating the problem as the field due to a circular current, we have to determine the scalar and vector-potentials.



Referring to the adjoining figure the scalar potential ϕ is given by

$$\phi = \frac{1}{4\pi}\int_0^{2\pi} a d\psi \cdot \frac{\eta}{\sqrt{(\rho - a\cos\psi)^2 + a^2\sin^2\psi + z^2}}$$

$$i.e.,\ \phi = \frac{a}{4\pi}\int_0^{2\pi} \frac{\eta d\psi}{\sqrt{\rho^2 + z^2 - 2\rho a\cos\psi + a^2}}$$

$$= \frac{\eta a}{4\pi}\int_{\pi/2}^{-\pi/2} \frac{-2d\psi_0}{\mathrm{R}\sqrt{1 - k^2\sin^2\psi_0}}\ ;\ \text{where } \mathrm{R}^2 = (\rho + a)^2 + z^2$$

$$k^2 = \frac{4\rho a}{\mathrm{R}^2}\ ;\ \psi_0 = \frac{\pi - \psi}{2}$$

$$= \frac{\eta a}{\pi \mathrm{R}}\int_0^{\pi/2} \frac{d\psi_0}{\sqrt{1 - k^2\sin^2\psi_0}}$$

$$(5,\ 1) \qquad \phi = a\eta\,\frac{\mathrm{K}(k)}{\pi \mathrm{R}} = \frac{e}{2\pi^2}\,\frac{\mathrm{K}(k)}{\mathrm{R}}$$

where K (k) is the complete elliptic integral of the first kind.

Coming to a question of units, we observe that for $\rho,\ z \gg a$, (4) reduces to $\phi \to \frac{e}{4\pi \mathrm{R}}$ since $\mathrm{K}(0) = \pi/2$

i.e., the units we are using are the ordinary electrostatic units and *not* the Gaussian units. To transform to the latter we need only replace e by $4\pi e$. This fact shall be made use of later.

Coming to the vector-potential due to the circular current,[6] we have, denoting this by $\vec{\mathrm{A}}$, $\mathrm{A}_\rho = 0$, $\mathrm{A}_z = 0$, and

$$(5,\ 2) \qquad \mathrm{A}_\psi = \frac{\eta v}{\pi}\sqrt{\frac{a}{\rho}} \cdot \frac{1}{k}\left\{\left(1 - \frac{k^2}{2}\right)\mathrm{K}(k) - \mathrm{E}(k)\right\}$$

where K (k) and E (k) are the complete elliptic integrals of the 1st and 2nd kind.

---

[6] See Debye, *Ency. Math. Wiss.*, V., 2, § 17, p. 434.

From the potentials ϕ and $\vec{A}$, the field strengths are given by

$$\vec{E} = -\operatorname{grad}\phi\,; \quad \vec{H} = \operatorname{rot}\vec{A}$$

$$\left.\begin{aligned} i.e.,\quad E_\rho &= -\frac{\partial\phi}{\partial\rho}\,;\; E_\psi = 0\,;\; E_z = -\frac{\partial\phi}{\partial z} \\ E_\rho &= -\frac{\partial A_\psi}{\partial z}\,;\; H_\psi = 0\,;\; H_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\psi) \end{aligned}\right\}$$

using (2, 5) and (2, 6)

From (4), we get

$$E_\rho = -\frac{\partial\phi}{\partial\rho} = \frac{a\eta}{\pi}\left[\frac{1}{R^2}\frac{\partial R}{\partial\rho}\cdot K(k) - \frac{1}{R}\cdot\frac{\partial K(k)}{\partial k}\cdot\frac{\partial k}{\partial\rho}\right]$$

$$R^2 = (\rho + a)^2 + Z^2\,;\; \frac{\partial R}{\partial\rho} = \frac{\rho + a}{R}\,;\; \frac{\partial R}{\partial z} = \frac{z}{R}$$

$$k^2 = \frac{4\rho a}{R^2}\,;\; \frac{1}{k}\frac{\partial R}{\partial\rho} = \frac{1}{2\rho} - \frac{\rho + a}{R^2}\,;\; \frac{1}{k}\frac{\partial k}{\partial z} = -\frac{z}{R^2}$$

From the theory of elliptic integrals,

$$\frac{\partial K(k)}{\partial k} = \frac{E(k)}{kk'^2} - \frac{K(k)}{k}$$

where k' is the modulus complementary to k, *i.e.*, $k'^2 = 1 - k^2$

$$i.e.,\quad k'^2 = \frac{r^2}{R^2} \quad \text{or} \quad \underline{k' = \frac{r}{R}}$$

Hence,

$$E_\rho = \frac{a\eta}{\pi}\left[\frac{1}{R^2}\cdot\frac{\rho + a}{R}K(k) - \frac{1}{R}\left\{\left(\frac{E(k)R^2}{r^2} - K(k)\right)\left(\frac{1}{2\rho} - \frac{\rho + a}{R^2}\right)\right\}\right]$$

(5, 3)
$$i.e.,\quad \left.\begin{aligned} E_\rho &= \frac{a\eta}{2\pi R\rho}\left\{K(k) - \frac{Z^2 + a^2 - \rho^2}{r^2}E(k)\right\} \\ E_z &= \frac{a\eta z}{\pi R r^2}E(k) \end{aligned}\right\}$$

Similarly

Coming now to the magnetic field strength, we get from (5, 2)

$$H_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\psi) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\frac{\eta v}{2\pi}\cdot R\left\{\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right\}\right]$$

$$H_z = \frac{\eta v}{2\pi\rho}\cdot\frac{\partial}{\partial\rho}\left[R\left\{\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right\}\right] \qquad \left[\frac{\partial E(k)}{\partial k} = \frac{E(k) - K(k)}{k}\right.$$

$$= \frac{\eta v}{2\pi\rho}\left[\frac{\partial R}{\partial\rho}\left\{\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right\} + R\left\{\left(1 - \frac{k^2}{2}\right)\frac{\partial K(k)}{\partial k}\cdot\frac{\partial k}{\partial\rho}\right.\right.$$

$$\left.\left. - K(k)\cdot k\frac{\partial k}{\partial\rho} - \frac{\partial E(k)}{\partial k}\cdot\frac{\partial k}{\partial\rho}\right\}\right]$$

$$= \frac{\eta v}{2\pi\rho}\left[\frac{\rho+a}{\mathrm{R}}\left\{\mathrm{K}(k) - \frac{\mathrm{E}(k)}{k'^2}\left(1-\frac{k^2}{2}\right)\right\} + \frac{\mathrm{R}}{2\rho}\left\{\frac{\mathrm{E}(k)}{k'^2}\left(1-\frac{k^2}{2}-k'^2\right) - \mathrm{K}(k)\,\frac{k^2}{2}\right\}\right]$$

Putting $k^2 = 4a\rho/\mathrm{R}^2$, $k'^2 = r^2/\mathrm{R}^2$, this reduces to

$$\mathrm{H}_z = \frac{\eta v}{2\pi\rho}\left[\frac{\rho}{\mathrm{R}}\,\mathrm{K}(k) + \mathrm{E}(k)\left\{\frac{a\mathrm{R}}{r^2} - \frac{(\rho+a)\,(\rho^2+a^2+\mathrm{z}^2)}{\mathrm{R}r^2}\right\}\right]$$

$$= \frac{\eta v}{2\pi\rho}\left[\frac{\rho}{\mathrm{R}}\,\mathrm{K}(k) - \frac{\mathrm{E}(k)\rho}{\mathrm{R}r^2}\,(\rho^2 - a^2 + z^2)\right]$$

$$= \frac{\eta v\rho}{2\pi\rho\mathrm{R}}\left\{\mathrm{K}(k) - \frac{\rho^2 - a^2 + z^2}{r^2}\,\mathrm{E}(k)\right\}$$

Similarly,

$$\mathrm{H}_\rho = -\frac{\partial \mathrm{A}_\psi}{\partial z} = -\frac{\partial}{\partial z}\left[\frac{\eta v}{2\pi}\cdot\frac{\mathrm{R}}{\rho}\left\{\left(1-\frac{k^2}{2}\right)\mathrm{K}(k) - \mathrm{E}(k)\right\}\right]$$

$$= -\frac{\eta v}{2\pi\rho}\,\frac{\partial}{\partial z}\left[\mathrm{R}\left\{\left(1-\frac{k^2}{2}\right)\mathrm{K}(k) - \mathrm{E}(k)\right\}\right]$$

$$= -\frac{\eta v}{2\pi\rho}\left[\frac{\partial \mathrm{R}}{\partial z}\left\{\left(1-\frac{k^2}{2}\right)\mathrm{K}(k) - \mathrm{E}(k)\right\} + \mathrm{R}\left\{\left(1-\frac{k^2}{2}\right)\frac{\partial \mathrm{K}(k)}{\partial k}\cdot\frac{\partial k}{\partial z} - k\mathrm{K}(k)\,\frac{\partial k}{\partial z} - \frac{\partial \mathrm{E}(k)}{\partial k}\cdot\frac{\partial k}{\partial z}\right\}\right]$$

$$= -\frac{\eta v}{2\pi\rho}\left[\frac{\mathrm{Z}}{\mathrm{R}}\left\{\mathrm{K}(k) - \frac{\mathrm{E}(k)}{k'^2}\left(1-\frac{k^2}{2}\right)\right\}\right]$$

simplifying by substituting the expressions for $\frac{\partial \mathrm{K}(k)}{\partial k}$ and $\frac{\partial \mathrm{E}(k)}{\partial k}$.

$$= -\frac{\eta v z}{2\pi\mathrm{R}\rho}\left\{\mathrm{K}(k) - \frac{\rho^2 + a^2 + \mathrm{Z}^2}{r^2}\,\mathrm{E}(k)\right\};$$

thus,

$$(5,\,4)\qquad \left\{\begin{aligned}\mathrm{H}_\rho &= -\frac{\eta v z}{2\pi\mathrm{R}\rho}\left\{\mathrm{K}(k) - \frac{\rho^2+a^2+z^2}{r^2}\,\mathrm{E}(k)\right\}\\ \mathrm{H}_z &= \frac{\eta v}{2\pi\mathrm{R}}\left\{\mathrm{K}(k) - \frac{\rho^2+a^2+z^2}{r^2}\right\}\mathrm{E}(k)\end{aligned}\right\}$$

In (5, 3) and (5, 4) we can transform the expressions

$$\frac{z^2+a^2-\rho^2}{r^2},\quad \frac{\rho^2+a^2+z^2}{r^2},\quad \frac{\rho^2-a^2+z^2}{r^2}$$

by writing $z^2 = r^2 - (\rho - a)^2$ and obtain

$$(5,\,5)\qquad \left\{\begin{aligned}\mathrm{E}_\rho &= \frac{a\eta}{\pi\mathrm{R}}\cdot\frac{\rho-a}{r^2}\,\mathrm{E}(k) + \frac{a\eta}{2\pi\mathrm{R}\rho}\,[\mathrm{K}(k) - \mathrm{E}(k)]\\ \mathrm{E}_z &= \frac{a\eta}{\pi\mathrm{R}}\cdot\frac{z}{r^2}\,\mathrm{E}(k)\end{aligned}\right\}$$

$$(5,\ 6)\qquad \left\{\begin{aligned} H_\rho &= \frac{a\eta v}{\pi R}\cdot\frac{z}{r^2}\,E(k) - \frac{\eta v z}{2\pi R\rho}\,[K(k) - E(k)] \\ H_z &= -\frac{a\eta v}{\pi R}\cdot\frac{\rho - a}{r^2}\,E(k) + \frac{\eta v}{2\pi R}\,[K(k) - E(k)] \end{aligned}\right\}$$

(5, 5) and (5, 6) are the exact classical solutions (where $\vec{B} = \vec{H}$ and $\vec{D} = \vec{E}$). Going to the limit $r \to 0$ which corresponds to the case $\rho = a$, $z = 0$, *i.e.*, on the ring, we have

$$R \to 2a,\ k' \to 0, \text{ and}$$

$$E(k) \to 1 \text{ and } K(k) \to \log (4/k').$$

Hence the classical solutions valid in the neighbourhood of the ring are

$$(5,\ 7)\qquad \left\{\begin{aligned} E_\rho &= \frac{\eta}{2\pi}\cdot\frac{\rho - a}{r^2} + \frac{\eta}{4\pi a}\left(\log\frac{8a}{r} - 1\right) \\ E_z &= \frac{\eta}{2\pi}\cdot\frac{Z}{r^2} \end{aligned}\right\}$$

and,

$$(5,\ 8)\qquad \left\{\begin{aligned} H_\rho &= \frac{\eta v}{2\pi}\cdot\frac{z}{r^2} \\ H_z &= -\frac{\eta v}{2\pi}\cdot\frac{\rho - a}{r^2} + \frac{\eta v}{4\pi a}\left(\log\frac{8a}{r} - 1\right) \end{aligned}\right\}$$

It is interesting to observe that the first terms in the right-hand members of (5, 7) and (5, 8) agree with the unitary field theory expression for the zero approximation case (4, 3), (4, 4), (4, 7), (4, 8) when, in the latter, we consider (r_1/r) small and neglect second and higher powers.

6. Method of procedure for calculating integrals of energy and angular momentum.

The remarks made above suggest the method of procedure to be adopted for the calculation of these integrals. We assume that the classical solutions hold good except in the immediate neighbourhood of the ring where we assume the field theory solutions (in §4) to be valid for values of r large compared with r_1. Thus, in the case of the total energy E, for example, we form the classical and field theory expressions U_{cl} and U_f and write

$$(6,\ 1)\qquad E = \int U_{cl}\,dV + U_f\,dV$$

Since we can certainly assume that most of the energy is concentrated in the neighbourhood of the ring we take the limits for r in the first integral r = a finite number $= f$ (say) for the upper limit and r for the lower limit such that *r is small compared with a*. The limit f is further so chosen that no terms appear after integration which $\to \infty$ as $r \to \infty$. For the second

integral the lower limit is taken 0 while the upper limit r is assumed *large compared with* r_1 but small compared with a. The angular momentum is similarly treated. Further in calculating U_f we use the expressions in §4 and for U_{el} the expressions (5, 7) – (5, 8), or, in case a higher approximation is needed, the expressions (5, 5) and (5, 6) with the substitution therein, up to the desired degree of accuracy, from the series—expansions for $K(k)$ and $E(k)$[7] valid in the neighbourhood of $k = 1$, or $k' = 0$, *viz.*,

$$(6, 2) \qquad \begin{cases} E(k) = 1 + \frac{k'^2}{2}\left(\log \frac{4}{k'} - \frac{1}{2}\right) + \frac{3}{16} k'^4 \left(\log \frac{4}{k'} - 1 - \frac{1}{12}\right) + \cdots \\ K(k) = \log \frac{4}{k'} + \frac{k'^2}{2}\left(\log \frac{4}{k'} - 1\right) + \cdots \end{cases}$$

7. Calculation of the energy.

$$U_f = L_f + (\vec{D} \cdot \vec{E})$$

$$= b^2 \left\{ \sqrt{1 + \frac{1}{b^2}(\vec{B}^2 - \vec{E}^2)} - 1 \right\} + \vec{D}\,\vec{E}, \text{ since } (\vec{B} \cdot \vec{E}) = 0 \text{ from } \S(4).$$

$$= b^2 \left\{ \sqrt{1 + \frac{1}{b^2}\left(\frac{\eta^2 v^2}{4\pi^2 r^2} \cdot \frac{r^2}{r^2 + r_1^2} - \frac{\eta^2}{4\pi^2 r^2} \cdot \frac{r^2}{r^2 + r_1^2}\right)} - 1 \right\} + \frac{\eta^2}{4\pi^2 r^2} \cdot \frac{1}{\sqrt{r^2 + r_1^2}}$$

$$= b^2 \left\{ \sqrt{1 - \frac{r_1^2}{r^2 + r_1^2}} - 1 \right\} + \frac{b^2 r_1^2}{a^2 r \sqrt{r^2 + r_1^2}}, \text{ using (4, 5)}$$

$$\therefore \int U_f \, dV = 2\pi \iint U_f \rho d\rho dz$$

$$= 2\pi \int_0^r \int_0^{2\pi} U_f (a + r \cos \phi) \, r dr d\phi$$

$$= 2\pi a \int_0^r \int_0^{2\pi} U_f \, r dr d\phi$$

since $\int_0^{2\pi} \cos \phi \cdot d\phi = 0$, and U_f is a function of r only.

We have here taken r and ϕ to be polar co-ordinates in the meridian plane, such that,

$$(7, 1) \qquad \begin{cases} \rho - a = r \cos \phi \\ z = r \sin \phi \\ d\rho dz = r dr d\phi \end{cases}$$

[7] See, for example, Schlomilch, *Komp. d. Hoh. Anal.*, Bd. 2, pp. 322–323.

$$i.e., \quad \int U_f \, dV = 4\pi^2 a \int_0^r U_f \, r dr$$

$$= 4\pi^2 \, ab^2 \int_0^r \left[\left\{ \sqrt{1 - \frac{r_1^2}{r^2 + r_1^2}} - 1 \right\} + \frac{r_1^2}{a^2 r \sqrt{r^2 + r_1^2}} \right] r dr$$

$$= 4\pi^2 \, ab^2 \int_0^r \left\{ \frac{r^2}{\sqrt{r^2 + r_1^2}} - r + \frac{r_1^2}{a^2 \sqrt{r^2 + r_1^2}} \right\} dr$$

$$= 4\pi^2 \, ab^2 \int_0^r \left(\sqrt{r^2 + r_1^2} - \frac{r_1^2}{\sqrt{r^2 + r_1^2}} - r \right) dr + 4\pi^2 \, ab'^2 \int_0^r \frac{r_1^2 \, dr}{\sqrt{r^2 + r_1^2}}$$

$$= 4\pi^2 \, ab^2 \int_0^r (\sqrt{r^2 + r_1^2} - r) \, dr + 4\pi^2 a \, (b'^2 - b^2) \int_0^r \frac{r_1^2 \, dr}{\sqrt{r^2 + r_1^2}}$$

$$= a\eta^2 \, \alpha^2 \int_0^x (\sqrt{1 + x^2} - x) \, dx + a\eta^2 v^2 \int_0^x \frac{dx}{\sqrt{1 + x^2}}$$

putting $r = r_1 x$ and using $r_1 = \frac{\alpha\eta}{2\pi b}$; $b' = \frac{b}{a}$.

$$= \tfrac{1}{2} \, a\eta^2 \alpha^2 \left\{ \frac{r}{r_1} \sqrt{1 + \frac{r^2}{r_1^2}} - \frac{r^2}{r_1^2} \right\} + \tfrac{1}{2} \, a\eta^2 \beta^2 \log \left\{ \frac{r}{r_1} + \sqrt{1 + \frac{r^2}{r_1^2}} \right\}$$

(7. 2) where $\beta^2 = 1 + v^2$

$$(7.\ 3) \qquad \int U_f \, dV = a\eta^2 \log \left(\frac{2r}{r_1} \right)$$

where we have made use of the fact that r is large compared with r_1 and that *v is very nearly equal to the velocity of light, i.e., unity.*

We now proceed to calculate the classical part of the energy and write, making use of (5, 7)-(5, 8),

$$U_{cl} = \tfrac{1}{2} (\vec{E}^2 + \vec{H}^2) = \tfrac{1}{2} (\vec{E}_\rho^2 + E_z^2 + H_\rho^2 + H_z^2)$$

$$= \frac{\eta^2 \beta^2}{8\pi^2 r^2} + \frac{\eta^2 \beta^2}{32\pi^2 a^2} \left(\log \frac{8a}{r} - 1 \right)^2 + \frac{\eta^2 \alpha^2}{8\pi^2 a} \cdot \frac{\rho - a}{r^2} \left(\log \frac{8a}{r} - 1 \right)$$

$$\therefore \int U_{cl} \, dV = \tfrac{1}{2} \, a\eta^2 \beta^2 \int_r^f \frac{dr}{r} + \frac{\eta^2 \beta^8}{8a} \int r \left(\log \frac{8a}{r} - 1 \right)^2 dr + \frac{\eta^2 \alpha^2}{8\pi^2 a}$$

$$\int\int \frac{\rho - a}{r^2} \left(\log \frac{8a}{r} - 1 \right) 2\pi \rho d\rho dz$$

$$\int\int \frac{\rho - a}{r^2}\left(\log\frac{8a}{r} - 1\right) 2\pi\rho d\rho dz = \int_r^f \int_0^{2\pi} \frac{r\cos\phi}{r^2}\left(\log\frac{8a}{r} - 1\right)$$

$$2\pi\,(a + r\cos\phi)\, r dr d\phi$$

$$= \int_r^f \int_0^{2\pi} 2\pi r \left(\log\frac{8a}{r} - 1\right)\cos^2\phi dr d\phi$$

$$= \int_r^f 2\pi^2 r\left(\log\frac{8a}{r} - 1\right) dr$$

Now,

$$\int r\left(\log\frac{8a}{r} - 1\right)^2 dr = \frac{r^2}{2}\left(\log\frac{8a}{r} - 1\right)^2 + \frac{r^2}{2}\left(\log\frac{8a}{r} - 1\right)$$

$$+ \frac{r^2}{4} = \frac{r^2}{2}\left(\log\frac{8a}{r}\right)^2 - \frac{r^2}{2}\log\frac{8a}{r} + \frac{r^2}{4}$$

and $\int r\left(\log\frac{8a}{r} - 1\right)dr = \frac{r^2}{2}\left(\log\frac{8a}{r} - 1\right) + \frac{r^2}{4} = \frac{r^2}{2}\left(\log\frac{8a}{r}\right) - \frac{r^2}{4}$

To avoid infinites at the upper limit we put $f = 8a$ and get

$$\int U_{cl}\, dV = \tfrac{1}{2}\, a\eta^2\beta^2 \log\frac{8a}{r} + 0\,(r\log r) + 0\,(r^2\log r) + 0\,(r^2).$$

Since $\underset{r\to 0}{\text{Lt}}\,(r\log r) = \underset{r\to 0}{\text{Lt}}\,(r^2\log r) = \underset{r\to 0}{\text{Lt}}\,(r^2) = 0$ and r is small compared with a, we can neglect the last three terms and write

$$(7, 4) \qquad \int U_{cl}\, dV = \tfrac{1}{2}\, a\eta^2\beta^2 \log\frac{8a}{r} = a\eta^2 \log\frac{8a}{r}$$

From (7, 3) and (7, 4), we get

$$E = a\eta^2 \log\left(\frac{2r}{r_1}\right) + a\eta^2 \log\left(\frac{8a}{r}\right)$$

$$(7, 5) \quad i.e., \; E = a\eta^2 \log\left(\frac{16a}{r_1}\right)$$

giving a finite value for the energy.

8. Calculation of the angular momentum.

Corresponding to (6, 1) we write

$$(8, 1) \qquad \vec{M} = \int \vec{M}_{cl} dV + \int \vec{M}_f\, dV.$$

$$\vec{M} = \int \{\vec{r} \times (\vec{D} \times \vec{B})\}\, dV$$

$$= \int \{\vec{D}\,(\vec{r}\,\vec{B}) - \vec{B}\,(\vec{r}\,\vec{D})\}\, dV$$

$$\therefore \quad \left.\begin{aligned} M_\rho &= \int \{Z\,(D_\rho B_z - B_\rho D_z)\}\, dV \\ M_z &= \int \{\rho\,(D_z B_\rho - B_z D_\rho)\}\, dV \\ M_\psi &= 0 \end{aligned}\right\}$$

From the fact of axial symmetry, or observing that the integrand in M_ρ changes sign when z is replaced by $-z$, we have $M_\rho = 0$ and $M = M_z$. Hence

$$\int M_f\, dV = \int \rho\,(D_z B_\rho - B_z D_\rho)\, dV$$

$$= \frac{\eta^2 v}{4\pi^2} \int \frac{\rho}{r\sqrt{r^2 + r_1^2}}\, dV$$

using (4, 3) and (4, 8)

$$i.e., \quad \int M_f\, dV = \frac{\eta^2 v}{4\pi^2} \int\int \frac{\rho \cdot 2\pi\rho d\rho dz}{r\sqrt{r^2 + r_1^2}}$$

$$= \frac{\eta^2 v}{2\pi} \int_0^r \int_0^{2\pi} \frac{(a + r\cos\phi)^2\, r dr d\phi}{r\sqrt{r^2 + r_1^2}}$$

$$= \frac{\eta^2 v}{2\pi} \int_0^r \frac{dr}{\sqrt{r^2 + r_1^2}}\,(2\pi a^2 + \pi r^2)$$

$$= \eta^2 v \int_0^r \frac{a^2 dr}{\sqrt{r^2 + r_1^2}} + \frac{\eta^2 v}{2} \int_0^r \frac{r^2 dr}{\sqrt{r^2 + r_1^2}}$$

$$= \eta^2 v a^2 \int_0^r \frac{dr}{\sqrt{r^2 + r_1^2}} + \frac{\eta^2 v}{2} \int_0^r \sqrt{r^2 + r_1^2}\, dr - \frac{\eta^2 v r_1^2}{2} \int_0^r \frac{dr}{\sqrt{r^2 + r_1^2}}$$

$$= \frac{\eta^2 v}{2} \int_0^r \sqrt{r^2 + r_1^2}\, dr + \frac{\eta^2 v}{2}\,(2a^2 - r_1^2) \int_0^r \frac{dr}{\sqrt{r^2 + r_1^2}}$$

$$= \frac{\eta^2 v r_1^2}{2} \int_0^x \sqrt{1 + x^2}\, dx + \frac{\eta^2 v}{2}\,(2a^2 - r_1^2) \int_0^x \frac{dx}{\sqrt{1 + x^2}}$$

$$= \frac{\eta^2 v r_1^2}{2}\,[\tfrac{1}{2} x\sqrt{1 + x^2} + \tfrac{1}{2}\log(x + \sqrt{1 + x^2})]$$

$$+ \frac{\eta^2 v}{2}\,(2a^2 - r_1^2)\log(x + \sqrt{1 + x^2})$$

$$= \frac{\eta^2 v}{2}\left(2a^2 - \frac{r_1^2}{2}\right)\log\left(\frac{r}{r_1} + \sqrt{1 + \frac{r^2}{r_1^2}}\right) + \frac{\eta^2 v r\sqrt{r^2 + r_1^2}}{4}$$

$$= \eta^2 v a^2 \log\left(\frac{r}{r_1} + \sqrt{1 + \frac{r^2}{r_1^2}}\right) + \frac{\eta^2 v}{4}\, r\sqrt{r^2 + r_1^2}$$

neglecting the term in r_1^2

$$(8,\ 2) \qquad \int \mathrm{M}_f\, d\mathrm{V} = \eta^2 v a^2 \log\left(\frac{2r}{r_1}\right) + \frac{\eta^2 v}{4}\, r^2$$

since r is assumed large compared with r_1.

On the classical theory.

$$\mathrm{M}_z = \int \rho\, (\mathrm{E}_z\, \mathrm{H}_\rho - \mathrm{H}_z\, \mathrm{E}_\rho)\, d\mathrm{V}$$

Using the exact expressions (5, 5)-(5, 6) we have

$$\mathrm{E}_z\mathrm{H}_\rho - \mathrm{H}_z\mathrm{E}_\rho = \frac{a\eta}{\pi \mathrm{R}} \cdot \frac{z}{r^2}\, \mathrm{E}(k) \left\{\frac{a\eta v}{\pi \mathrm{R}} \cdot \frac{z}{r^2}\, \mathrm{E}(k) - \frac{\eta v z}{2\pi \mathrm{R}\rho}\, [\mathrm{K}(k) - \mathrm{E}(k)]\right\}$$

$$- \left\{\frac{a\eta}{\pi \mathrm{R}} \cdot \frac{\rho - a}{r^2}\, \mathrm{E}(k) + \frac{a\eta}{2\pi \mathrm{R}\rho}\Big(\mathrm{K}(k) - \mathrm{E}(k)\Big)\right\} \left\{-\frac{a\eta v}{\pi \mathrm{R}} \cdot \frac{\rho - a}{r^2}\, \mathrm{E}(k) + \frac{\eta v}{2\pi \mathrm{R}}\, \mathrm{K}(k) - \mathrm{E}(k)\right\}$$

$$= \frac{a\eta^2 v}{\pi^2 \mathrm{R}^2}\left[\left\{a\mathrm{E}^2 \frac{z^2}{r^4} - \frac{1}{2\rho} \cdot \frac{z^2}{r^2}\, \mathrm{E}(\mathrm{K} - \mathrm{E})\right\} + \left\{\left(\frac{\rho - a}{r^2}\, \mathrm{E} + \frac{1}{2\rho}\, \overline{\mathrm{K} - \mathrm{E}}\right)\left(\frac{a}{r^2}\, \overline{\rho - a}\, \mathrm{E} - \tfrac{1}{2}\, \overline{\mathrm{K} - \mathrm{E}}\right)\right\}\right]$$

$$= \frac{a\eta^2 v}{\pi^2 \mathrm{R}^2}\left[\left\{a\mathrm{E}^2 \frac{z^2}{r^4} - \frac{1}{2\rho} \frac{z^2}{r^2}\, \mathrm{E}\, (\mathrm{K} - \mathrm{E})\right\} + \left\{a\mathrm{E}^2 \frac{(\rho - a)^2}{r^4} + \mathrm{E}\, (\mathrm{K} - \mathrm{E}) \left(\frac{a}{2\rho} \cdot \frac{\rho - a}{r^3} - \frac{\rho - a}{2r^2}\right) - \frac{1}{4\rho}\, (\mathrm{K} - \mathrm{E})^2\right\}\right]$$

$$= \frac{a\eta^2 v}{\pi^2 \mathrm{R}^2}\left\{\frac{a\mathrm{E}^2}{r^2} - \frac{\mathrm{E}(\mathrm{K} - \mathrm{E})}{2\rho} - \frac{1}{4\rho}\, (\mathrm{K} - \mathrm{E})^2\right\}$$

$$= \frac{a\eta^2 v}{\pi^2 \mathrm{R}^2}\left\{\frac{a\mathrm{E}^2}{r^2} + \frac{\mathrm{E}^2}{2\rho} - \frac{\mathrm{EK}}{2\rho} + \frac{\mathrm{EK}}{2\rho} - \frac{\mathrm{K}^2}{4\rho} - \frac{\mathrm{E}^2}{4\rho}\right\}$$

$$= \frac{a\eta^2 v}{\pi^2 \mathrm{R}^2}\left\{\left(\frac{a}{r^2} + \frac{1}{4\rho}\right)\mathrm{E}^2 - \frac{1}{4\rho}\, \mathrm{K}^2\right\}$$

$$= \frac{a\eta^2 v}{4\pi^2 \rho \mathrm{R}^2}\left\{\frac{\mathrm{R}^2}{r^2}\, \mathrm{E}^2 - \mathrm{K}^2\right\}$$

$$(8,\ 3) \quad \therefore \quad \rho\, (\mathrm{E}_z\mathrm{H}_\rho - \mathrm{H}_z\, \mathrm{E}_\rho) = \frac{a\eta^2 v}{4\pi^2 \mathrm{R}^2}\left\{\frac{[\mathrm{E}(k)]^2}{k'^2} - [\mathrm{K}(k)]^2\right\}$$

Before proceeding further with the approximations for $\mathrm{E}\,(k)$ and $\mathrm{K}\,(k)$ valid in the neighbourhood of the ring, we might notice the very elegant form (8, 3) for the angular-momentum, this being the exact expression. It might also be observed that (8, 3) shows directly that $\mathrm{M}\rho$ is an odd function of z, for, the expression

$$\mathrm{E}_\rho\mathrm{H}_z - \mathrm{H}_\rho\mathrm{E}_z = \frac{a\eta^2 v}{4\pi^2 \mathrm{R}^2}\left\{[\mathrm{K}(k)]^2 - \frac{[\mathrm{E}(k)]^2}{k'^2}\right\}$$

is an even function of z, since only squares of E (k), K (k), k' and R appear in it and further k, k', r and R are all even functions of Z. Hence the expression

$$Z\,(E_\rho H_z - H_\rho E_z) \text{ is an odd function of Z.}$$

We now proceed with the approximation of (8, 3) by writing

$$E\,(k) = 1 + \frac{k'^2}{2}\left(\log\frac{4}{k'} - \tfrac{1}{2}\right) + \frac{3}{16}\,k'^4\left(\log\frac{4}{k'} - 1 - \frac{1}{12}\right)$$

and $\quad K\,(k) = \log\frac{4}{k'} + \frac{k'^2}{4}\left(\log\frac{4}{k'} - 1\right)$, and get

$$\frac{E^2}{k'^2} - K^2 = \frac{1}{k'^2} + \left\{\log\frac{4}{k'} - \tfrac{1}{2} - \left(\log\frac{4}{k'}\right)^2\right\} + k'^2\left\{-\tfrac{1}{4}\left(\log\frac{4}{k'}\right)^2 + \frac{5}{8}\log\frac{4}{k'} - \frac{11}{32}\right\}$$

and this does not depend on ϕ; hence

$$\int M_{cl}\,dV = \int \rho\,(E_z H_\rho - H_z E_\rho)\,dV$$

$$= a^2\eta^2 v \int_r^f \frac{1}{R^2}\left(\frac{R^2}{r^2}\,[E(k)]^2 - [K(k)]^2\right) r\,dr$$

$$= a^2\eta^2 v \int \left\{\frac{[E(k)]^2}{r^2} - \frac{[K(k)]^2}{R^2}\right\} r\,dr$$

Substituting the series-expansions for E (k), K (k) and $R = 2a$ and proceeding with the integration as in the case of the energy we take $f = 8a$. We neglect as before, the terms $(r \log r)$ and $(r^2 \log r)$ at the lower limit but retain the term in r^2, in view of the form of expression (8, 2). This gives

$$(8, 4) \qquad \int M_{cl}\,dV = a^2\eta^2 v \log\left(\frac{8a}{r}\right) - \frac{\eta^2 v}{4}\,r^2$$

From (8, 2) and (8, 4) we get easily

$$(8, 5) \qquad M = a^2\eta^2 v \log\,(16a/r_1)$$

giving *a finite value for the angular-momentum as well.*

[A simpler way of deducing (8, 5) would have been to neglect the terms in r^2 both in (8, 2) and (8, 4) since in both cases r^2 appears as an absolute term.]

9. *Magnetic moment.*

The magnetic moment can be calculated from the expression (5, 2) for the classical vector-potential, *viz.*,

$$A_\psi = \frac{\eta v}{\pi}\sqrt{\frac{a}{\rho}}\cdot\frac{1}{k}\left\{\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right\}$$

For values of $k \ll 1$, *i.e.*, at great distance from the ring,[8]

$$\begin{aligned} A_\psi &= \frac{\eta v}{32}\sqrt{\frac{a}{\rho}} \cdot R^3 \left\{1 + \frac{3}{4}k^2 + - -\right\} \\ &= \frac{\eta v}{32}\sqrt{\frac{a}{\rho}} \cdot \frac{8\sqrt{a\rho}\cdot a\rho}{R^3}\left\{1 + \frac{3}{4}\cdot\frac{4\rho a}{R^2} + - -\right\} \\ &= \frac{a^2\eta v}{4} \cdot \frac{\rho}{R^3}\left\{1 + \frac{3\rho a}{R^2} + - -\right\} \end{aligned}$$

Hence the magnetic moment

$$(9,\ 1) \qquad m = \frac{a^2\eta v}{4} = \frac{a^2 e v}{8\pi a} = \frac{aev}{8\pi}$$

10. Spin.

We shall write the expressions for the energy, angular momentum and magnetic moment in Gaussian units since we have so far used only electrostatic units (also $c = 1$). This requires as remarked on p. 363, the replacing of e by $4\pi e$ *i.e.*, η by $4\pi\eta$.

$$(10,\ 1) \qquad E = a\eta^2 \log\left(\frac{16a}{r_1}\right) = \frac{ae^2}{4\pi^2 a^2}\log\left(\frac{16a}{r_1}\right) \longrightarrow \frac{4e^2}{a}\log\left(\frac{16a}{r_1}\right)$$

$$(10,\ 2) \qquad M = a^2\eta^2 v \log\left(\frac{16a}{r_1}\right) = \frac{e^2 v}{4\pi^2}\log\left(\frac{16a}{r_1}\right) \longrightarrow 4e^2 v \log\left(\frac{16a}{r_1}\right)$$

$$(10,\ 3) \qquad m = \frac{a^2\eta v}{4} = \frac{a^2 e v}{8\pi a} = \frac{aev}{8\pi} \longrightarrow \frac{aev}{2}$$

Hence $$\frac{E}{M} = \frac{1}{av} = \frac{e}{2m} \text{ and,}$$

putting $E = \mu c^2$ we have $E = \mu$ since $c = 1$, which gives

$$(10,\ 4) \qquad \frac{m}{M} = \frac{e}{2\mu}$$

showing that if the particle represented by the ring-singularity be an electron, *the spin of the electron is not explained* by our classical method of investigation.

11. The mass of the proton.

If we consider the ring-singularity to represent the proton it can be shown that we can derive an estimate of the ring-radius which would give the right proton mass.

Equating (10, 2) to $\hbar = h/2\pi$, we get

$$\frac{h}{2\pi} = 4e^2 v \log\left(\frac{16a}{r_1}\right)$$

---

[8] *Cf.* Debye, *ibid.*, p. 435.

where r_1 is given from (4, 5), using Gaussian-units, as

$$r_1 = \frac{ae}{\pi ab} = \sqrt{1-v^2} \cdot \frac{e}{\pi ba}$$

Introducing now the radius r_0 of the point charge of the field theory,[9] which is given by $e = br_0^2$, we have

$$r_1 = \sqrt{1-v^2}\ \frac{r_0^2}{\pi a}$$

and

$$\frac{16a}{r_1} = \frac{16\pi a^2}{r_0^2} \cdot \frac{1}{\sqrt{1-v^2}} = 16\pi \left(\frac{a}{r_0}\right)^2 \frac{1}{\sqrt{1-v^2}}$$

and (10, 2) gives

$$\frac{h}{2\pi} = 4e^2 v \log \left\{ 16\pi \left(\frac{a}{r_0}\right)^2 \cdot \frac{1}{\sqrt{1-v^2}} \right\}$$

Introducing c, instead of taking $c = 1$, this can be written in the form

$$(11, 1) \quad \frac{\hbar c}{e^2} = \frac{4v}{c} \log \left\{ 16\pi \left(\frac{a}{r_0}\right)^2 \frac{1}{\sqrt{1-v^2/c^2}} \right\}$$

We can similarly write the expression for the energy (10, 1) in the form

$$(11, 2) \quad \mathrm{E} = \frac{4e^2}{a} \log \left\{ 16\pi \left(\frac{a}{r_0}\right)^2 \frac{1}{\sqrt{1-v^2/c^2}} \right\}$$

Let m_0 = mass of a point charge and M_0 = mass of the ring, which we now assume to represent the mass of the proton. We have[10]

$$(11, 3) \quad 1{\cdot}236\ \frac{e^2}{r_0} = m_0 c^2$$

Putting $\mathrm{E} = \mathrm{M}_0 c^2$ in (11, 2) and making use of (11, 3) we get

$$(11, 4) \quad \frac{\mathrm{M}_0}{m_0} = \frac{\mathrm{E}}{1{\cdot}236 e^2/r_0} = \frac{4r_0}{1{\cdot}236a} \log \left[\frac{16\pi}{a} \left(\frac{a}{r_0}\right)^2 \right]$$

From the equations (11, 1) and (11, 4) we can determine v/c and r_0/a if we observe that the left-hand side of (11, 1) is the fine structure constant whose value we shall take as 137 and that the left-hand side of (11, 4) is the ratio of the masses of proton and electron which we shall take as 1840. Thus

$$\frac{4r_0}{1{\cdot}236a} \log \left[\frac{16\pi}{a} \left(\frac{a}{r_0}\right)^2 \right] = 1840 \text{ from (11, 4), or}$$

$$(11, 5) \quad \frac{4r_0}{a} \log \left[\frac{16\pi}{a} \left(\frac{a}{r_0}\right)^2 \right] = 2274, \text{ and}$$

$$(11, 6) \quad \frac{4v}{c} \log \left[\frac{16\pi}{a} \left(\frac{a}{r_0}\right)^2 \right] = 137$$

---

[9] Born and Infeld, *Proc. Roy. Soc.*, A, 1934, **144**, p. 439.

[10] *Ibid.*, p. 446, equation (8, 7).

We can solve the equations (11, 5) and (11, 6) for v/c and r_0/a. Denoting these by x and y, the above equations can be written in the form

$$\left.\begin{aligned} &(11, 7)\quad 4x \log\left(\frac{16\pi}{x^2\sqrt{1-y^2}}\right) = 2274 \\ &(11, 8)\quad 4y \log\left(\frac{16\pi}{x^2\sqrt{1-y^2}}\right) = 137 \end{aligned}\right\}$$

The second equation can be written in the form (using only the nearest integers)

$$\frac{16\pi}{x^2\sqrt{1-y^2}} = \epsilon^{34/y} \quad (\epsilon \text{ is the exponential})$$

$$(11, 9)\quad x^2 = \frac{16\pi}{\sqrt{1-y^2}}\,\epsilon^{-34/y}$$

Dividing (11, 8) by (11, 7), we have

$$(11, 10)\quad \frac{y}{x} = \frac{137}{2274} \text{ or } y = \frac{x}{17}$$

Hence eliminating x between (11, 9) and (11, 10)

$$y = \frac{4\sqrt{\pi}}{17} \cdot \frac{1}{(1-y^2)^{\frac{1}{4}}}\,\epsilon^{-17/y}$$

$$(11, 11)\quad i.e.,\ y\,(1-y^2)^{\frac{1}{4}} = \frac{8}{11}\,\epsilon^{-17/y}$$

From this equation we can estimate how near the value of y is to unity. Putting $1 - y^2 = \epsilon_0^4$ (ϵ_0 an infinitesimal) the order of magnitude of ϵ_0 is determined by

$$\epsilon_0\,(1-\epsilon_0^4)^{\frac{1}{2}} = \frac{8}{11}\,\epsilon^{-17} \quad \epsilon_0^4 \sim \epsilon^{-68} \text{ or } (1 - v^2/c^2) \sim {}^{-68} \sim 10^{-29}$$

$$(11, 12)\quad i.e.,\ \epsilon_0 \sim \epsilon^{-17} \text{ or } v/c \sim 1$$

showing that $y = v/c$ is *very nearly equal to unity*. From equation (11, 10) we can therefore write

$$(11, 13)\quad x = r_0/a \sim 17.$$

This gives an *estimate of the ring-radius*[11] *adequate to explain the mass of the proton.*

---

[11] Dr. Beth has kindly written to me about an attempt to determine the value of a for which the energy has a minimum. For this purpose, he takes account of some additional terms which I have omitted in (7, 4) and in place of (7, 5) he takes:

$$E = \frac{4e^2}{a} \cdot \frac{1+v^2}{2} \log\left[\frac{16\pi}{a}\left(\frac{a}{r_0}\right)^2\right].$$

The minimising of E gives rise to a quintic equation which he has solved by a method of approximation, giving

$$a = r_0 \cdot 10^{16 \pm 1,6}$$

a result which, as Prof. Born has pointed out to me in a letter, is of an impossible order of magnitude.

12. Conclusion.

I hope to return in a future paper to the treatment of the ring-singularity in the general case and deal with questions connected with the conservation laws and self force and the equations of motion in a constant external field.

It gives me the greatest pleasure to thank Prof. Max Born for suggesting the problem, constantly guiding and advising me while here and corresponding with me from Cambridge.

BANGALORE
PRESS
LAKE VIEW
MYSORE RD
BANGALORE
CITY